1. Define uniform convergence of a sequence of function (f_n) defined on $A \subseteq R$ to R. If $A \subseteq R$ and $\phi : A \to R$ then define the uniform norm of ϕ on A. Discuss the uniform and point wise convergence of the sequence (f_n), where

$$f_n(x) = \frac{x}{n} \text{ for } x \in R \text{ and } n \in N.$$

Soln -

A sequence of functions (f_n) defined on $A \subseteq R$ to R is said to be converge uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$, if for each $\varepsilon > 0$, there exist a natural number m, such that whenever $n \geq m$.

$|f_n(x) - f(x)| < \varepsilon$ for uniform convergence

$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in A_0$

It is denoted by $f_n \rightrightarrows f$ on A_0.

Uniform Norm -

Let ϕ be a bounded function defined on a non-empty subset $A \subseteq \mathbb{R}$ to $\mathbb{R}$. then the uniform norm of ϕ on A is defined as.

$$\|\phi\|_A = \sup\{\phi(x) \mid x \in A.$$

$$f_n(x) = \frac{x}{n} \text{ for } x \in R. \text{ and } n \in N.$$

$$\lim \left(\frac{x}{n}\right) = 0 \ \forall \ x \in \mathbb{R}$$

$\therefore$ (f_n) converges to f.

pointwise where $f(x) = 0 \ \forall \ x \in \mathbb{R}$.

By M_n- Test. $\|f_n - f\| : x \in R$

$$= \sup \{ f_n(x) - f|$$

$$= \sup \left| \frac{1}{n} - 0 \right| = \frac{1}{n}$$

$$= \lim \left(\frac{1}{n}\right)$$

$$= \frac{1}{n}$$

$$= 0. \text{ Converges to zero.}$$

(f_n) converges uniformly to f.

Ques-b. let (f_n) be the sequence of functions defined by $f_n(x) = \frac{1}{1+x^n} \ \forall \ x \in [0,1], n \in \mathbb{N}$

find the point wise limit of the sequence $f(n)$. Does (f_n) converges uniformly? Justify your answer.

Ans- $f_n(x) = \frac{1}{1+x^n}$

$$f_n(x) = \begin{cases} 1, & x \leq 1 \\ 1/2, & x = 1 \\ 0 & x > 1 \end{cases}$$

$$\|f_n - f\| = \sup |f_n(x) - f(x)| \, , \, x \in [0,1]$$

$$\sup \left\{ \left| \frac{1}{1+x^n} - 0 \right| , n \in [0,1) \right\} \cup \sup \left\{ \frac{1}{1+x^n} - 1 \right| ; x=1 \right\}.$$

$$\sup \left[\left\{ \frac{1}{1+x^n} : n \in [0,1] \right\} \cup \left\{ 1/2 \right\} \right.$$

$$\sup \left[\left\{ \frac{1}{1+x^n} : n \in [0,1] \right\} \cup \left\{ 1/2 \right\} \right.$$

$$\sup \{ 1, 1/2 \}$$

$$= 1/2$$

By M_n- Test the Sequence is not Convergent.

C. Show that if (f_n) and $g(n)$ are two Sequences of bounded functions on $A \subseteq R$ to R that converges uniformly to f and g respectively then prove that the product Sequence $(f_n g_n)$ converge uniformly on A to fg. Give an example to show that in general the product of two uniformly convergent Sequence may not be uniformly convergent.

Soln - since (f_n) and (g_n) are uniformly convergent Sequences of bounded functions, they are uniformly bounded by "lemma" Every uniformly convergent Sequences of bounded functions is uniformly bounded.

Therefore, there exist $M, N > 0$ such that

$$|f_n(x)| \leq M \text{ and } |g_n(x)| \leq N$$
$$\forall\, x \in A.$$

Again lemma f and g are bounded functions
Hence there exist $P_1, P_2 > 0$ such that

$$|f(x)| \leq P_1 \text{ and } |g(x)| < P_2 \;\; \forall\, x \in A.$$

We write $K = \max\{k_1, k_2\}$

Now,

$$|(f_n g_n)(x) - (fg)(x)|$$
$$= |f_n(x) g_n(x) - f(x) g(x)|$$
$$= |f_n(x) g_n(x) - g_n(x) f(x) + g_n(x) f(x) - f(x) g(x)|$$
$$\leq |g_n(x) f_n(x) - g_n(x) f(x)| + |g_n(x) f(x) - f(x) g(x)|$$
$$= |g_n(x)|\,|f_n(x) - f(x)| + |f(x)|\,|g_n(x) - g(x)|$$
$$\leq K \cdot \frac{\varepsilon}{2K} + K \cdot \frac{\varepsilon}{2K} = \varepsilon$$

for all $n \geq k$ and for all $x \in A$.

Hence $(f_n g_n)$ converges uniformly to fg on A.

Que-2. (a). Let (f_n) be a sequence of integrable functions on $[a,b]$ and Suppose that (f_n) converges uniformly to f on $[a,b]$. show that f is integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Sol^n ⇒ $\left(\int_a^b f_n dx\right)$ is Convergent. Since (f_n) is uniformly

Step I - convergent it Satisfies Cauchy Criterion. Hence for $\varepsilon > 0$. $\exists$ $p \in \mathbb{N}$

Such that

$$|f_n(x) - f_m(x)| < \frac{b}{b-a} \quad \forall\ n,m > p$$

$$\Rightarrow \frac{-\varepsilon}{b-a} < f_n(x) - f_m(x) < \frac{\varepsilon}{b-a}$$

$$\Rightarrow \frac{-\varepsilon}{b-a} \int_a^b 1 dx < \int_a^b (f_n(x) - f_m(x)) dx < \frac{\varepsilon}{b-a} \int_a^b 1 dx$$

$$-\varepsilon < \int_a^b f_n dx - \int_a^b f_m dx < \varepsilon$$

$$\left| \int_a^b f_n dx - \int_a^b f_m dx \right| < \varepsilon \ \forall\ n,m \geq p$$

The Sequence $\left(\int_a^b f_n dx\right)$ is Cauchy Sequence.

Hence it is Convergent. we write,

$$\lim_{n\to\infty} \left(\int_a^b f_n dx\right) = A.$$

step II — $f \in R[a,b]$, $\int_a^b f dx = A$

As, (f_n) converges uniformly to f, for $\varepsilon > 0$
$\exists$ $k_1 \in \mathbb{N}$ such that

$|f_n(x) - f(x)| < \varepsilon \; \forall \; n \geq k_1, \forall \; x \in [a,b]$ —— (I)

As $\left(\int_a^b f_n dx\right)$ converges to A, given $\varepsilon > 0$,
$\exists$ $k_2 \in \mathbb{N}$ such that

$\left|\int_a^b f_n dx - A\right| < \varepsilon \; \forall \; n \geq k_2$ —— (II).

Let $k = \max\{k_1, k_2\}$, then f_k satisfies both (I) and (II). Also, as f_k is integrable given $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$\left|\int f_k dx - S(f_k, P)\right| < \varepsilon$
$\forall$ partitions P with $\|P\| < \delta$. —— (III).

Let $P = \{x = a, x_1, \ldots x_n = b\}$ be a partition with $\|P\| < \delta$, then for $t_i \in [x_{i-1}, x_i]$

We have $|S(f, P) - S(f_k, P)|$

$= \left|\sum_{i=1}^{n} \left(f(t_i)(x_i - x_{i-1}) - f_k(t_i)(x_i - x_{i-1}\right)\right|$.

$< \varepsilon \sum |(x_i - x_{i-1})|$ by — (I).

$\therefore |S(f,P) - S(f_k,P)| < \varepsilon(b-a)$ — (iv)

$\therefore |S(f,P) - A| = |S(f,P) - S(f_k,P) + S(f_k,P) - \int_a^b f_k\,dx + \int_a^b f_k\,dx - A|$

$\leq |S(f,P) - S(f_k,P)| + |S(f_k,P) - \int_a^b f_k\,dx| + |\int_a^b f_k\,dx - A|$

$< \varepsilon(b-a) + \varepsilon + \varepsilon$ using (i), (ii), (iii) & (iv)

$= \varepsilon(2+b-a)$

As $\varepsilon > 0$ is arbitrary, we have

$$\lim_{\|P\| \to 0} S(f,P) = A$$

$\Rightarrow f$ is integrable, and $\int_a^b f\,dx = A$.

Ques-6. Let $f_n(x) = \frac{x^n}{n}$ for $x \in [0,1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on $[0,1]$ and that the sequence (f_n') converges on $[0,1]$ to a function g but the convergence is not uniform.

(12). Soln - $f_n(x) = \frac{x^n}{n}$ defined on $[0,1]$ then,
the sequence is convergent pointwise on $[0,1]$ to the function f defined by

$$f(x) = \lim \left(\frac{x^n}{n}\right) = 0$$

$$|f_n(x) - f(x)| = \left|\frac{x^n}{n} - 0\right|$$

$$= \left|\frac{x^n}{n}\right| \leq \frac{1}{n}$$

$$< \varepsilon \text{ if } n > \frac{1}{\varepsilon}$$

Choose a natural number k such that $k > \frac{1}{\varepsilon}$. then, for given $\varepsilon > 0$, we have number k, such that whenever $n \geq k$

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in [0,1]$$

Hence the convergence is uniform.

Now, consider the derivative $g_n(x) = f'_n(x)$

$$= \frac{n x^{n-1}}{n} = x^{n-1} \text{ on } [0,1]$$

Here the sequence (g_n) is convergent pointwise to the function g defined by $g(x) = \lim g_n(x)$

$$= \lim (x^{n-1}) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$$

limit function is not continuous but all the g_n's are continuous on $[0,1]$.
therefore, the convergence of (f_n') is not uniform on $[0,1]$.

C. Show that the sequence $\left(\frac{x^n}{1+x^n}\right)$ does not converge uniformly on $[0,2]$.

Soln -

$$\lim_{n\to\infty}(f_n(x)) = \lim_{n\to\infty}\left(\frac{x^n}{1+x^n}\right)$$

$$= \begin{cases} 0, & x \in 0 \\ \frac{1}{2}, & x = 1 \\ 1, & 1 < x \leq 2 \end{cases}$$

$\therefore (f_n)$ converges to f, when $f(x) = 0 \ \forall \ x \in (0,2]$.

by Mn Test $\|f_n - f\|$

$= \sup\{|f_n - f| \ \forall \ x \in (0,2)\}$

$= \sup\{|$

Now, $\int_0^2 f_n(x)\,dx = \int_0^2 \frac{x^n}{1+x^n}\,dx$

$=$

at

ons

Sol^n - The given sequence of functions $\left(\frac{x^n}{1+x^n}\right)$ is convergent pointwise to the limit function f defined by

$$f(x) \begin{cases} 0 & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } 1 < x \le 2 \end{cases}$$

Here, the limit function f is discontinuous at $x = 1$ but each f_n is continuous on $[0,2]$. Therefore the convergence is not uniform.

Ques-3. State and prove Weierstrass M-Test for uniform convergence of series function.

Statement - let Σf_n be a series defined on $A \subseteq R$ let (M_n) be a sequence of positive real number such that

$$|f_n(x)| \le M_n \; \forall \; x \in A, \forall \; n \in N$$

If ΣM_n converges, then Σf_n converges uniformly.

proof - let ΣM_n be convergent, then it satisfies Cauchy criterion of convergence of series for $\varepsilon > 0$, $\exists$ $K \in N$ such that

$$|M_{m+1} + M_{m+2} + M_{m+3} - - - + M_n| < \varepsilon$$

$\forall \; n \ge m \ge K$

$M_{m+1} + M_{m+2} + M_{m+3} \cdots + M_n < \varepsilon \ \forall\ n \geq m$,

now,

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots f_n(x)|$$

$$\leq |f_{m+1}(x)| + |f_{m+2}(x)| + \cdots |f_n(x)|$$

$$< M_{m+1} + M_{m+2} + \cdots + M_n < \varepsilon$$

$$\forall\ n \geq m \geq k$$

$\therefore$ $\sum f_n$ satisfies Cauchy criterion

$\sum f_n$ is convergent. proved

(b). Show that the series $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ is uniformly convergent on $[-a, a]$, $a > 0$ but is not uniformly convergent on $\mathbb{R}$.

Soln - Hence $f_n(x) = \sin\left(\frac{x}{n^2}\right)$

$$f_n(x) = \left|\sin \frac{x}{n^2}\right|$$

$$\leq \left|\frac{x}{n^2}\right| \quad \because \sin x \leq |x|$$

We choose, $a > 0$.

Case I - $|x| \leq a$, then we have

$$|f_n(x)| \leq \frac{a}{n^2}$$

The Series $\sum \frac{1}{n^2}$ is Convergent by p-test $(p = 2>1)$

Hence $\sum f_n(x) = \sum \sin\left(\frac{x}{n^2}\right)$ is uniformly convergent, by M-Test.

Case II - $|x| > a$, we take

$$x_k = k^2, \quad n_k = k$$

then
$$|f_{n_k}(x_k)| = |f_k(k^2)|$$
$$= \left|\sin\left(\frac{k^2}{k^2}\right)\right|$$
$$= |\sin 1| \geq \varepsilon_0.$$

where $\varepsilon_0 = |\sin 1|$

$\therefore \sum f_n$ is not uniformly convergent.

Q. Discuss the pointwise Convergence of the Series of functions $\sum_{n=1}^{\infty} \frac{x^n}{2+3x^n}$ for $x \geq 0$.

Soln -

$$f_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{2+3x^n}$$
$$= \begin{cases} \frac{1}{5}, & x \geq 1 \\ 0, & x = 0 \end{cases}$$

$\therefore$ the Series is pointwise convergent.

Ques - 4. (a). Let f_n be continuous function on $D \subseteq \mathbb{R}$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on D. Prove that f is continuous on D. (C)

proof ⇒ We defined (S_n), the sequence of partial sums of $\sum f_n$ by

$$S_n = f_1 + f_2 + \cdots + f_n$$

Thus S_n is continuous on A as sum of continuous function is continuous

now, $\sum f_n$ converges uniformly to f

⇒ (S_n) converges uniformly to f.

⇒ f is continuous

Since $(S_n) \rightrightarrows f$ & each f_n is cts

f is cts.

(b). Show that the series of functions $\sum_{n=1}^{\infty} \frac{\cos(x^2+1)}{n^3}$ converges uniformly on $\mathbb{R}$ to a continuous function.

Soln - Here $f_n(x) = \frac{\cos(x^2+1)}{n^3}$

$$\leq \frac{1}{n^3}$$

for $n \geq 1$ $\sum \frac{1}{n^3}$ is convergent (by p-test)

∴ $\sum f_n$ is uniformly convergent.

3. Given the Riemann integrable functions

$f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, such that

$$f_n(x) = \sin\left(\frac{x}{n^2}\right) \text{ for all } x \in \mathbb{R}$$

Show that $\displaystyle\sum_{n=1}^{\infty} \int_{-2\pi}^{2\pi} f_n(x)\,dx = \int_{-2\pi}^{2\pi} \sum_{n=1}^{\infty} f_n(x)\,dx$

Que.-5. Define the radius of convergence and interval of convergence of power series. Check the uniform convergence of the following power series [-1,1]

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \cdots$$

Soln - Defn $\Rightarrow$ Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. If the sequence $(|a_n|^{1/n})$ is bounded, we set

$$\rho = \lim\sup(|a_n|^{1/n}).$$

If the sequence $(|a_n|^{1/n})$ is not a bounded sequence, then we set $\rho = \infty$.

The radius of convergence of $\sum a_n x^n$, to be denoted by R, is defined by

$$R = \begin{cases} 0, & \text{if } \rho = \infty \\ 1/\rho, & \text{if } 0 < \rho < \infty \\ \infty, & \text{if } \rho = 0 \end{cases}$$

The interval $(-R, R)$ is called the interval of convergence

Soln.

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} \cdots\cdots \quad [-1,1]$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Here the series of functions Σf_n is uniformly convergent for all. $x \in \mathbb{R}$ (by Leibnitz Test). Also we have.

$$\left|\frac{x^k}{k^2}\right| \le \frac{1}{k^2}$$

for $x \in [-1,1]$. Thus by the Weierstrass M-Test, the series of function Σf_n is convergent uniformly on $[-1,1]$.

b. state and prove the Cauchy Hadamard Theorem.

Statement - let $\Sigma a_n x^n$ be a power Series and $\mathbb{R}$ be its radius of Convergence.

(I). $0 < R < \infty$, then $\Sigma a_n x^n$ is absolutely Convergent iff $|x| < R$ and divergent if $|x| > R$

(II). If $R = 0$, then $\Sigma a_n x^n$ is Convergent only at $x = 0$.

(III). If $R = \infty$, then $\Sigma a_n x^n$ is absolutely convergent for all x.

proof - let x be any fixed real number then $\Sigma a_n x^n$ is a series of real number is

Case - (I). $0 < R < \infty$

By Cauchy root test, $\Sigma |a_n x^n|$ is Convergent if $\limsup \left(|a_n x_n|^{1/n}\right) < 1$ and divergent if $\limsup \left(|a_n x^n|^{1/n}\right) > 1$

now, $\lim \left(|a_n x^n|^{1/n} \right)$

$|x| \left(\limsup |a_n|^{1/n} \right)$

$|x| \rho$

$\dfrac{|x|}{R}$

$\therefore \sum a_n x^n$ is absolutely convergent if $\dfrac{|x|}{R} < 1$, that is if $|x| < R$, and $\sum a_n x_n$ is divergent if $\dfrac{|x|}{R} > 1$, that is

if $|x| > R$

Case II: $R = 0$ In this case,

$\limsup \left(|a_n x^n|^{1/n} \right) = \infty > 1$
$|x| \neq 0$

$\therefore$ The series $\sum a_n x^n$ is divergent for all. $x \neq 0$.

Case III - $R = \infty$

In this case,

$\limsup \left(|a_n x^n|^{1/n} \right) = 0 < 1$

$\therefore$ The series is absolutely convergent for all x.

(a) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then prove that the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ has also radius of convergence R and for $|x| < R$.

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

proof - $x \in (0, R)$, let $S_n(x)$ denote the sequence of partial sums for the series $\sum a_n x^n$

that is $S_n(x) = \sum_{k=0}^{n} a_k x^k$

$$f(x) = \lim_{n\to\infty} (S_n(x))$$

$$\Rightarrow \int_0^x f(t)\,dt = \int_0^x \lim_{n\to\infty} (S_n(t)\,dt$$

$$= \lim_{n\to\infty} \int_0^x (S_n(t)\,dt$$

($\because$ since (S_n) converges uniformly to f in $[0, x]$

$$= \lim_{n\to\infty} \int_0^x \left(\sum_{k=0}^{n} a_k t^k \right) dt$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

proved.

Similarly we can prove for $x \in (-R, 0)$.

Que-6. ⓐ. for Cosine function $C(x)$ and Sine function $S(x)$ prove the following:

(i) If $f: R \to R$ is such that $f''(x) = -f(x)$ for $x \in R$ then there Exist real numbers α and β such that $f(x) = \alpha C(x) + \beta S(x)$ for $x \in R$.

(ii). If $x \in R, x \geq 0$ then

$$1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

proof - (i).

(b). State Abell's theoram, show that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - - - \quad -1 \le x \le 1$$

and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - - -$

Statement - Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be power series with finite positive radius of Convergence R, if the series Converges at $x = R$ then if is Continuous at $x = R$, if the series Converges at $x = -R$, then f is Continuous at $x = -R$

Soln - We know that

$$1 + x + x^2 + - - - = \frac{1}{1-x} \quad |x| \le 1$$

$$\Rightarrow 1 + (-x^2) + (-x^2)^2 + - - - = \frac{1}{1+x^2}$$

$$\Rightarrow 1 - x^2 + x^4 - x^6 + - - - = \frac{1}{1+x^2}$$

Intergrating term by term, we get

$$\int_0^x (1 - t^2 + t^4 - t^6 + - -)dt = \int_0^x \frac{1}{1+t^2} dt.$$

$$\Rightarrow x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - - - = \tan^{-1} x, \; |x| < 1 -$$

At $x = 1$ - The seris becomes

$$1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + - - -$$ which is

alternating series and is convergent by Leibnitz test then by Ables theoram

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \tan^{-1}(1)$$
$$= \frac{-\pi}{4}.$$

(c). Prove that for every continuous function f on $[0,1]$; the sequence of polynomials $B_n f \to f$ uniformly on $[0,1]$, where $(B_n f)$ is the sequence of Bernstein's polynomials for the function f.